

PARABOLIC JOHN-NIRENBERG SPACES

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ABSTRACT. We introduce a parabolic version of John-Nirenberg space with exponent p and show that it is contained in local weak- L^p spaces.

1. INTRODUCTION

In the classical paper of F. John and L. Nirenberg [10], where functions of bounded mean oscillation (BMO) were introduced, they also studied a class satisfying a weaker BMO type condition

$$K_f^p := \sup_{\{Q_j\}_j} \sum_j |Q_j| \left(\int_{Q_j} |f - f_{Q_j}| dx \right)^p < \infty,$$

where the supremum is taken over all partitions $\{Q_j\}_j$ of a given cube Q_0 into pairwise non-overlapping subcubes. The functional $f \mapsto K_f$ defines a seminorm and the class of functions satisfying $K_f < \infty$, which we denote by $JN_p(Q_0)$ for John-Nirenberg space with exponent p , can be seen as a generalization of BMO. Indeed, BMO is obtained as the limit case of JN_p in the sense that

$$\lim_{p \rightarrow \infty} K_f = \sup_{Q \subseteq Q_0} \int_Q |f - f_Q| dx = \|f\|_{\text{BMO}(Q_0)}.$$

In contrast to the exponential integrability of BMO functions, $K_f < \infty$ implies that f belongs to the space weak- $L^p(Q_0)$. This was already observed by John and Nirenberg. Precisely, they showed that for $\lambda > 0$, we have

$$|\{x \in Q_0 : |f(x) - f_{Q_0}| > \lambda\}| \leq C \left(\frac{K_f}{\lambda} \right)^p,$$

where the constant C depends on n and p . Simpler proofs and generalizations have appeared in [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14]. In this

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note we show that an analogous result holds in the context of parabolic BMO spaces.

2. PARABOLIC JOHN-NIRENBERG SPACE

We shall introduce some notation and terminology. Given an Euclidean cube $Q = \prod_{i=1}^n [a_i, a_i + h]$, we define the forward in time translation

$$Q^+ := \prod_{i=1}^{n-1} [a_i, a_i + h] \times [a_n + h, a_n + 2h].$$

Moreover, we use the notation $Q^{+,2} := (Q^+)^+$. We write $f \in \text{BMO}^+(\mathbb{R}^n)$, if we have

$$(2.1) \quad \|f\|_{\text{BMO}^+(\mathbb{R}^n)} := \sup_Q \int_Q (f - f_{Q^+})^+ dx < \infty,$$

where the supremum is taken over all cubes in \mathbb{R}^n with sides parallel to the coordinate axes. It should be observed that despite the notation, the quantity defined by (2.1) is not actually a norm.

The one-dimensional $\text{BMO}^+(\mathbb{R})$ class was first introduced by F. J. Martín-Reyes and A. de la Torre [13], who showed that this class possesses many properties similar to the standard BMO space. Even though steps towards a multidimensional theory has been taken (see [4]), a satisfactory theory has only been developed in dimension one. In the classical elliptic setting, one of the cornerstones of theory of BMO functions is the celebrated John-Nirenberg inequality, which shows that logarithmic growth is the maximum possible for a BMO function. A corresponding result holds for the class $\text{BMO}^+(\mathbb{R})$, and a slightly weaker version of this result for $\text{BMO}^+(\mathbb{R}^n)$ was obtained in [4].

In this setting we define John-Nirenberg spaces as follows. We write $f \in JN_p^+(\mathbb{R}^n)$ if

$$(2.2) \quad (K_f^+)^p := \sup_{\{Q_j\}_j} \sum_j |Q_j| \left(\int_{Q_j \cup Q_j^+} (f - f_{Q_j^{+,2}})^+ dx \right)^p < \infty,$$

where the supremum is taken over countable families $\{Q_j\}$ of pairwise non-overlapping cubes satisfying $\sum_j |Q_j| < \infty$. The definition is reasonable in the sense that the $\text{BMO}^+(\mathbb{R}^n)$ condition may be seen as the limit case of (2.2) as $p \rightarrow \infty$. Precisely,

$$\lim_{p \rightarrow \infty} K_f^+ = \sup_Q \int_{Q \cup Q^+} (f - f_{Q^{+,2}})^+ dx,$$

where the quantity on the right-hand side is equivalent (up to a multiplication by a universal constant) to the BMO^+ norm of f , defined by (2.1).

The following theorem is a parabolic version of the weak distribution inequality of John and Nirenberg.

Theorem. *Assume $f \in JN_p^+(\mathbb{R}^n)$. Then, for every cube Q_0 and $\lambda > 0$, we have*

$$(2.3) \quad |\{x \in Q_0 : (f(x) - f_{Q_0^{+,2}})^+ > \lambda\}| \leq C \left(\frac{K_f^+}{\lambda} \right)^p,$$

where C only depends on n and p .

3. PROOF OF THE THEOREM

We follow the argument used in [1]. Given a non-negative f and a cube Q_0 , denote by $\Delta = \Delta(Q_0)$ the family of all dyadic subcubes obtained from Q_0 by repeatedly bisecting the sides into two parts of equal length. We shall make use of the “forward in time dyadic maximal function” defined by

$$M_{Q_0}^{+,d}f(x) := \sup_{\substack{Q \in \Delta \\ x \in Q}} \int_{Q^+} f \, dx.$$

A standard stopping-time argument shows that we have

$$\{x \in Q_0 : M_{Q_0}^{+,d}f(x) > \lambda\} = \bigcup_j Q_j,$$

where Q_j ’s are the maximal dyadic subcubes of Q_0 satisfying

$$(3.1) \quad \int_{Q_j^+} f \, dx > \lambda.$$

Maximality implies that the cubes Q_j are pairwise non-overlapping. Moreover, if $\lambda \geq f_{Q_0^+}$, then Q_0 doesn’t satisfy (3.1). Consequently, in this case every Q_j is contained in a larger dyadic subcube Q_{j-} of Q_0 which does not satisfy (3.1). Since $Q_j^{+,2} \subset Q_{j-}^+$, we conclude

$$(3.2) \quad \int_{Q_j^{+,2}} f \, dx \leq 2^n \lambda,$$

provided $\lambda \geq f_{Q_0^+}$. Standard arguments imply a weak type estimate for $M_{Q_0}^{+,d}$. Indeed, we have

$$|\{x \in Q_0 : M_{Q_0}^{+,d} f(x) > \lambda\}| = \sum_j |Q_j|.$$

While the cubes Q_j are non-overlapping, the cubes Q_j^+ may not be. Let us replace $\{Q_j^+\}_j$ by the maximal non-overlapping subfamily $\{\tilde{Q}_j^+\}_j$ which we form by collecting those Q_j^+ which are not properly contained in any other $Q_{j'}^+$. Maximality of $\{\tilde{Q}_j^+\}_j$ enables us to partition the family $\{Q_j\}_j$ as follows. Given \tilde{Q}_j^+ , we define $I_j := \{i : Q_i^+ \subseteq \tilde{Q}_j^+\}$, and we may write $\{Q_j\}_j = \bigcup_j \{Q_i : i \in I_j\}$. Now, whenever $i \in I_j$, we have $Q_i \subseteq \tilde{Q}_j \cup \tilde{Q}_j^+$ and we get the estimate

$$\begin{aligned} \sum_j |Q_j| &= \sum_j \sum_{i \in I_j} |Q_i| \\ &\leq 2 \sum_j |\tilde{Q}_j^+| \\ &\leq \frac{2}{\lambda} \int_{Q_0 \cup Q_0^+} f \, dx. \end{aligned}$$

Combining the previous estimates, we arrive at

$$(3.3) \quad |\{x \in Q_0 : M_{Q_0}^{+,d} f(x) > \lambda\}| \leq \frac{2}{\lambda} \int_{Q_0 \cup Q_0^+} f \, dx.$$

We begin by proving the following good λ inequality for the forward in time dyadic maximal operator.

Lemma. *Assume $f \in JN_p^+(\mathbb{R}^n)$ and take $0 < b < 2^{-n}$. Then, given a cube Q_0 , we have*

$$\begin{aligned} &|\{x \in Q_0 : M_{Q_0}^{+,d}(f - f_{Q_0^{+,2}})^+(x) > \lambda\}| \\ &\leq \frac{aK_f^+}{\lambda} |\{x \in Q_0 : M_{Q_0}^{+,d}(f - f_{Q_0^{+,2}})^+(x) > b\lambda\}|^{1/q}, \end{aligned}$$

whenever

$$b\lambda \geq \int_{Q_0^+} (f - f_{Q_0^{+,2}})^+ \, dx.$$

Here $a = 4(1 - 2^n b)^{-1}$ and q is the conjugate exponent of p .

Proof. Setting

$$E_Q(\lambda) := \{x \in Q : M_Q^{+,d}(f - f_{Q_0^{+,2}})^+(x) > \lambda\},$$

we may write the statement as

$$(3.4) \quad |E_{Q_0}(\lambda)| \leq \frac{4K_f^+}{(1 - 2^n b)\lambda} |E_{Q_0}(b\lambda)|^{1/q}.$$

Consider the function $(f - f_{Q_0^{+,2}})^+$ and form the decomposition as above at level $b\lambda$ to obtain a family of pairwise non-overlapping dyadic sub-cubes with

$$E_{Q_0}(b\lambda) = \bigcup_j Q_j.$$

Since $b\lambda < \lambda$, we have $E_{Q_0}(\lambda) \subset E_{Q_0}(b\lambda)$. It now follows that

$$(3.5) \quad E_{Q_0}(\lambda) = \bigcup_j E_{Q_j}(\lambda).$$

We claim that for every j ,

$$(3.6) \quad |E_{Q_j}(\lambda)| \leq \frac{2}{(1 - 2^n b)\lambda} \int_{Q_j \cup Q_j^+} (f - f_{Q_j^{+,2}})^+ dx.$$

Consider the functions $g_j := (f - f_{Q_j^{+,2}})^+$. To prove (3.6) it suffices to show that

$$(3.7) \quad E_{Q_j}(\lambda) \subset \{x \in Q_j : M_{Q_j}^{+,d} g_j(x) > (1 - 2^n b)\lambda\}.$$

Indeed, (3.6) then follows at once from the weak type estimate (3.3) applied to the functions g_j with λ replaced by $(1 - 2^n b)\lambda$. Let $x \in E_{Q_j}(\lambda)$ for some j . Then there exists a dyadic subcube Q of Q_j containing x and satisfying

$$\int_{Q^+} (f - f_{Q_0^{+,2}})^+ > \lambda$$

From (3.2) we have

$$\int_{Q_j^{+,2}} (f - f_{Q_0^{+,2}})^+ \leq 2^n b\lambda.$$

Combining these, we obtain

$$\begin{aligned}
(1 - 2^n b)\lambda &< \int_{Q^+} (f - f_{Q_0^{+,2}})^+ dx - \int_{Q_j^{+,2}} (f - f_{Q_0^{+,2}})^+ dx \\
&\leq \int_{Q^+} (f - f_{Q_0^{+,2}})^+ dx - \left(\int_{Q_j^{+,2}} f - f_{Q_0^{+,2}} dx \right)^+ \\
&= \int_{Q^+} (f - f_{Q_0^{+,2}})^+ - (f_{Q_j^{+,2}} - f_{Q_0^{+,2}})^+ dx \\
&\leq \int_{Q^+} (f - f_{Q_j^{+,2}})^+ dx \\
&\leq M_{Q_j}^{+,d} g_j(x).
\end{aligned}$$

Having now seen that (3.6) holds, we use (3.5) and sum over all j to obtain

$$\begin{aligned}
|E_{Q_0}(\lambda)| &= \sum_j |E_{Q_j}(\lambda)| \\
&\leq \frac{2}{(1 - 2^n b)\lambda} \sum_j \int_{Q_j \cup Q_j^+} (f - f_{Q_j^{+,2}})^+ dx \\
&= \frac{2}{(1 - 2^n b)\lambda} \sum_j |Q_j|^{1/q} |Q_j|^{-1/q} \int_{Q_j \cup Q_j^+} (f - f_{Q_j^{+,2}})^+ dx \\
&\leq \frac{4K_f^+}{(1 - 2^n b)\lambda} \left(\sum_j |Q_j| \right)^{1/q},
\end{aligned}$$

where the last inequality follows from the Hölder inequality and the definition of K_f^+ . Remembering also that $E_Q(b\lambda) = \bigcup_j Q_j$, we obtain the desired estimate. \square

We now complete the proof of the theorem by iterating the previous lemma. Except for a few details, this is just a repetition of the argument used in [1].

Proof of the Theorem. Using the same notation as in the proof of the lemma, we shall show

$$(3.8) \quad |E_{Q_0}(\lambda)| \leq C \left(\frac{K_f^+}{\lambda} \right)^p.$$

Let us choose

$$\lambda_0 := \frac{2K_f^+}{b|Q_0|^{1/p}}$$

and assume $\lambda > \lambda_0$. Then take $N \in \mathbb{Z}_+$ such that

$$(3.9) \quad b^{-N}\lambda_0 \leq \lambda < b^{-(N+1)}\lambda_0 = \frac{2b^{-(N+2)}K_f^+}{|Q_0|^{1/p}}.$$

By the definition of K_f^+ , we have

$$(3.10) \quad \frac{1}{|Q_0|} \int_{Q_0 \cup Q_0^+} (f - f_{Q_0^{+,2}})^+ dx \leq \frac{2K_f^+}{|Q_0|^{1/p}} = b\lambda_0.$$

In particular, this implies

$$\frac{1}{b} \int_{Q_0^+} (f - f_{Q_0^{+,2}})^+ dx \leq \lambda_0 \leq b^{-1}\lambda_0 \leq \dots \leq b^{-N}\lambda_0,$$

allowing us to apply the previous lemma successively N times to estimate the left-hand side of (3.8) as follows:

$$\begin{aligned} & |E_{Q_0}(\lambda)| \\ & \leq |E_{Q_0}(b^{-N}\lambda_0)| \\ & \leq \frac{aK_f^+}{b^{-N}\lambda_0} \cdot \left(\frac{aK_f^+}{b^{-N+1}\lambda_0} \right)^{1/q} \cdot \dots \cdot \left(\frac{aK_f^+}{b^{-1}\lambda_0} \right)^{1/q^{N-1}} |E_{Q_0}(\lambda_0)|^{1/q^N} \\ & \leq \frac{aK_f^+}{b\lambda} \cdot \left(\frac{aK_f^+}{b^2\lambda} \right)^{1/q} \cdot \dots \cdot \left(\frac{aK_f^+}{b^N\lambda} \right)^{1/q^{N-1}} \cdot \left(\frac{2}{\lambda_0} \int_{Q_0 \cup Q_0^+} (f - f_{Q_0^{+,2}})^+ dx \right)^{1/q^N}, \end{aligned}$$

where the last inequality follows from the weak type estimate (3.3) and the first inequality in (3.9). By the choice of λ_0 and (3.10) we further estimate

$$\begin{aligned} |E_{Q_0}(\lambda)| & \leq \left(\frac{aK_f^+}{\lambda} \right)^{1+q^{-1}+\dots+q^{-(N-1)}} \cdot b^{-(1+2q^{-1}+\dots+Nq^{-(N-1)})} \cdot (2b|Q_0|)^{1/q^N} \\ & = \left(\frac{aK_f^+}{\lambda} \right)^{p-p/q^N} \cdot b^{-(1+2q^{-1}+\dots+Nq^{-(N-1)})+q^{-N}} \cdot 2^{1/q^N} \cdot |Q_0|^{1/q^N}. \end{aligned}$$

Since both $1 + 2q^{-1} + \dots + Nq^{-(N-1)}$ and $p - p/q^N$ remain bounded as $N \rightarrow \infty$, we have

$$|E_{Q_0}(\lambda)| \leq C|Q_0|^{1/q^N} \left(\frac{1}{\lambda} \right)^{p-p/q^N}.$$

Finally, we notice that from the second inequality in (3.9) we get

$$|Q_0|^{1/q^N} \left(\frac{1}{\lambda}\right)^{-p/q^N} = \lambda^{p/q^N} |Q_0|^{1/q^N} \leq 2^{p/q^N} b^{-(N+2)p/q^N} \leq C$$

with C independent of N . Thus we have arrived at the desired estimate.

For $0 < \lambda \leq \lambda_0$ we use the trivial estimate

$$|\{x \in Q_0 : (f(x) - f_{Q_0^{+,2}})^+ > \lambda\}| \leq |Q_0| = \frac{2^p (K_f^+)^p}{b^p \lambda_0^p} \leq C \left(\frac{K_f^+}{\lambda}\right)^p.$$

□

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